# CS 237: Probability in Computing

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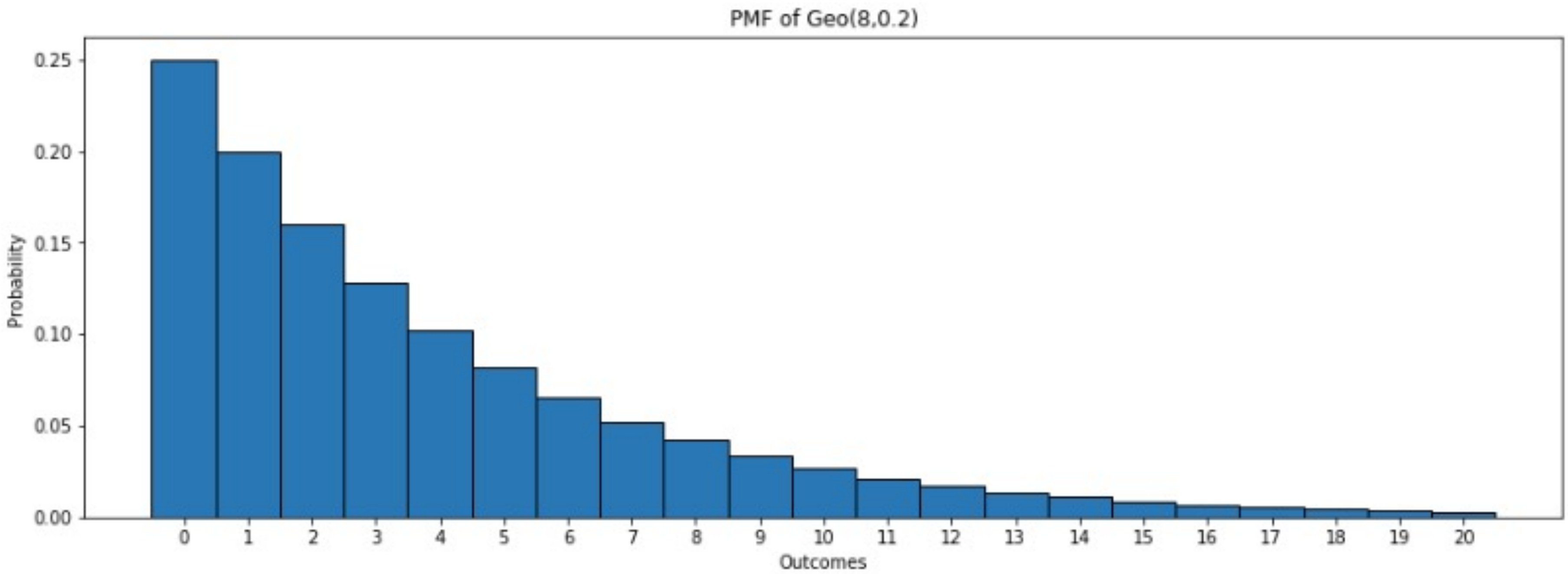
## Lecture 20:

- Exponential Distribution
- Poisson Process
- Poisson (Discrete) Distribution

# Exponential Distribution as Limit of Geometric



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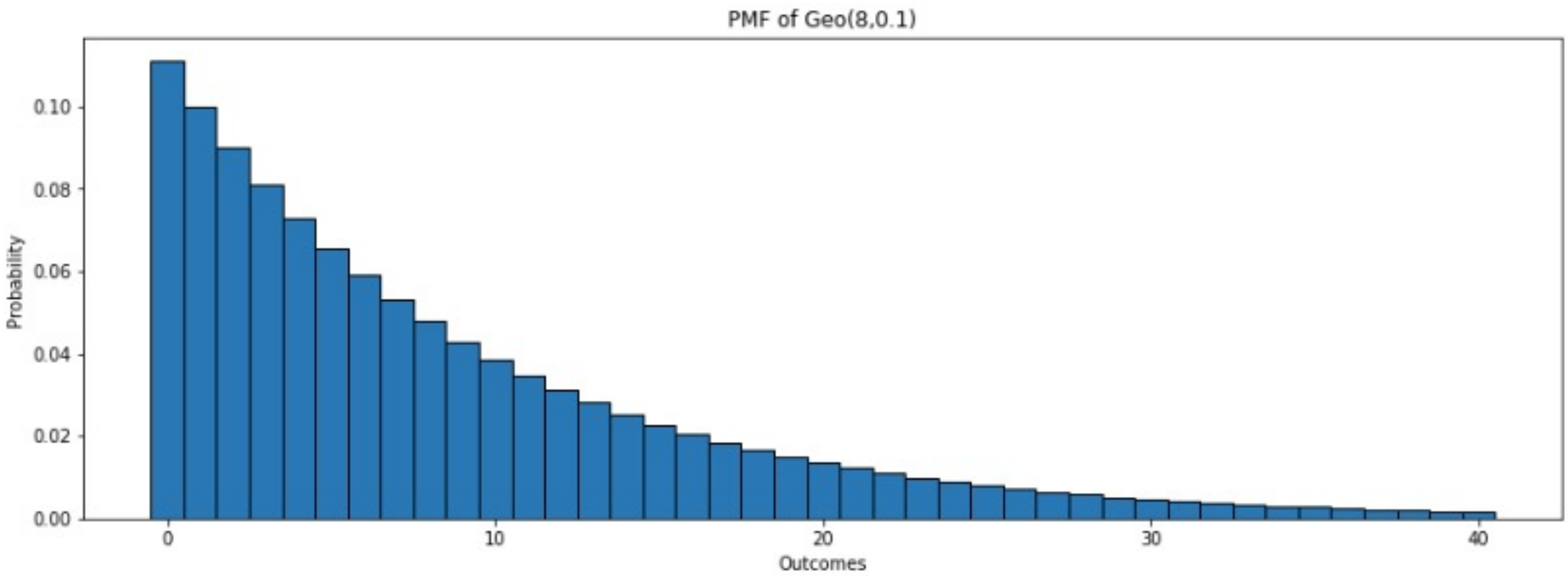

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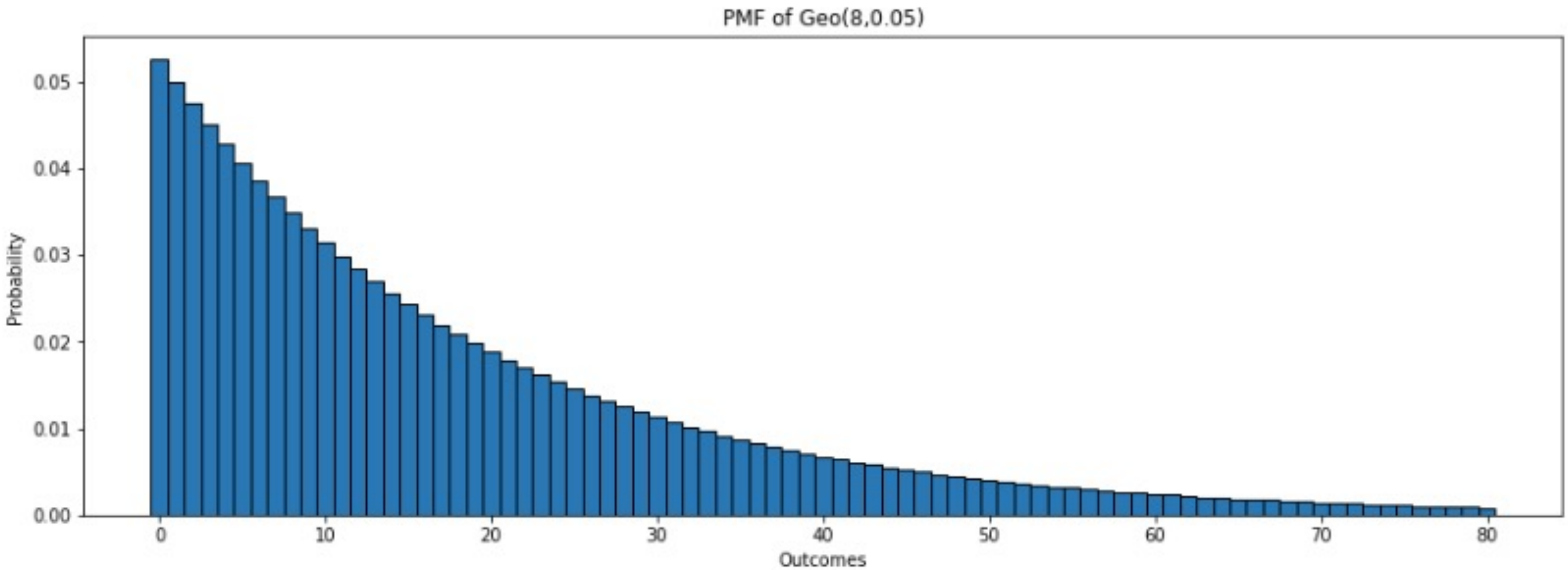

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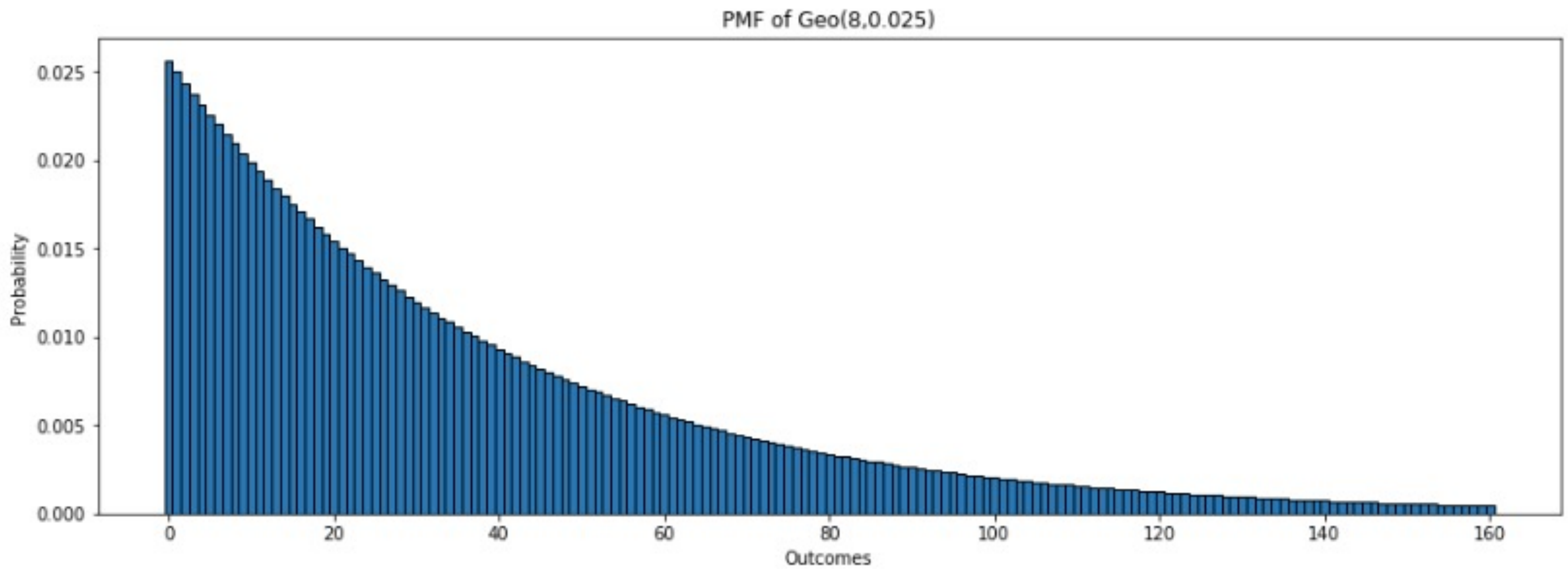

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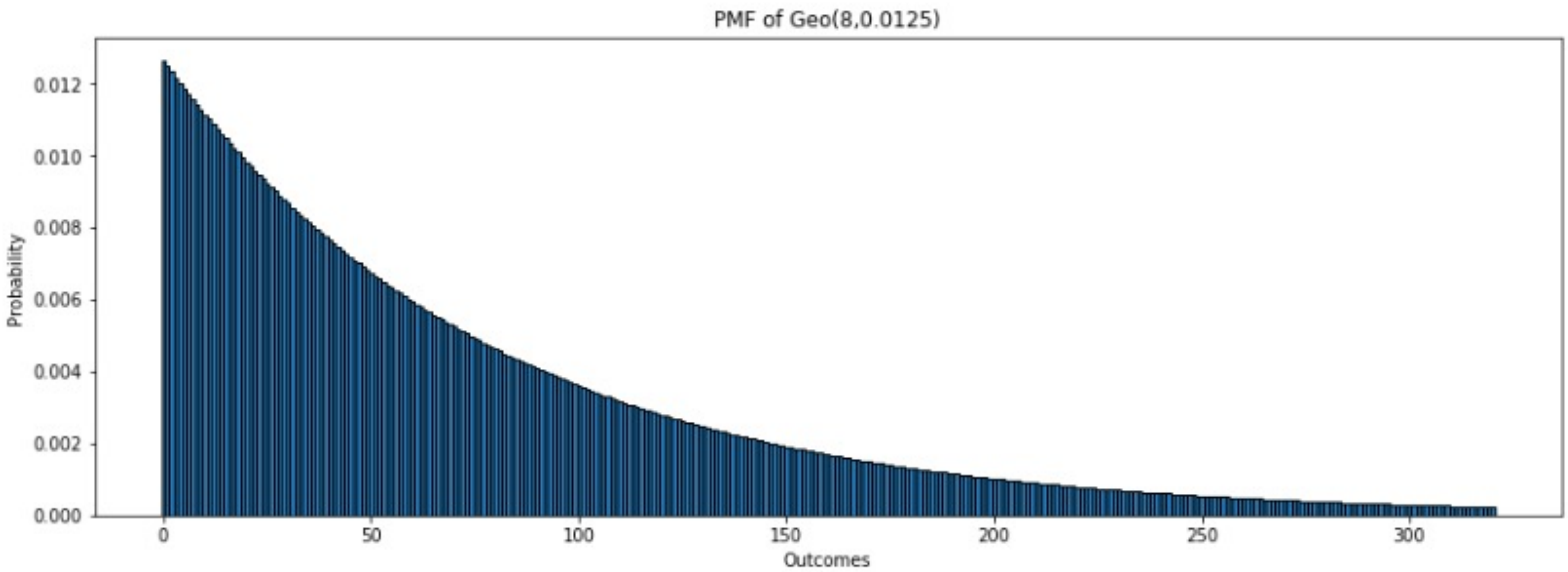

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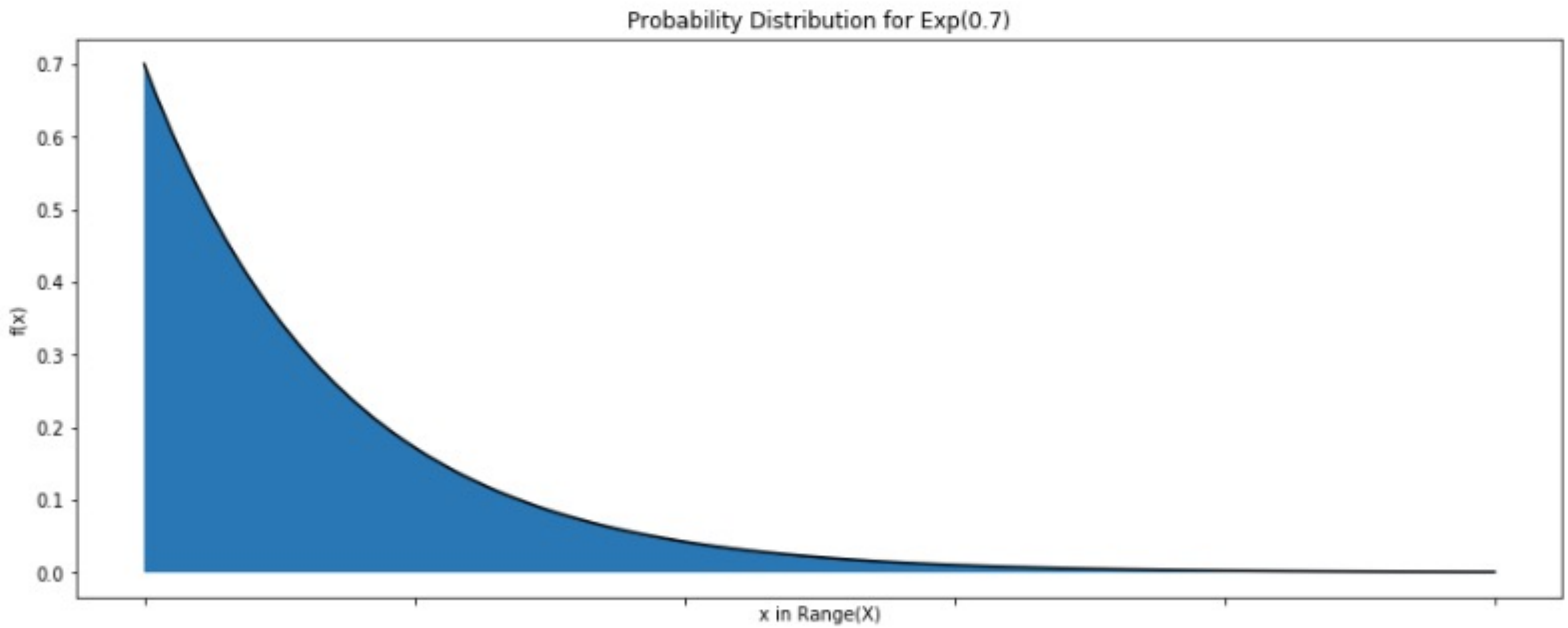
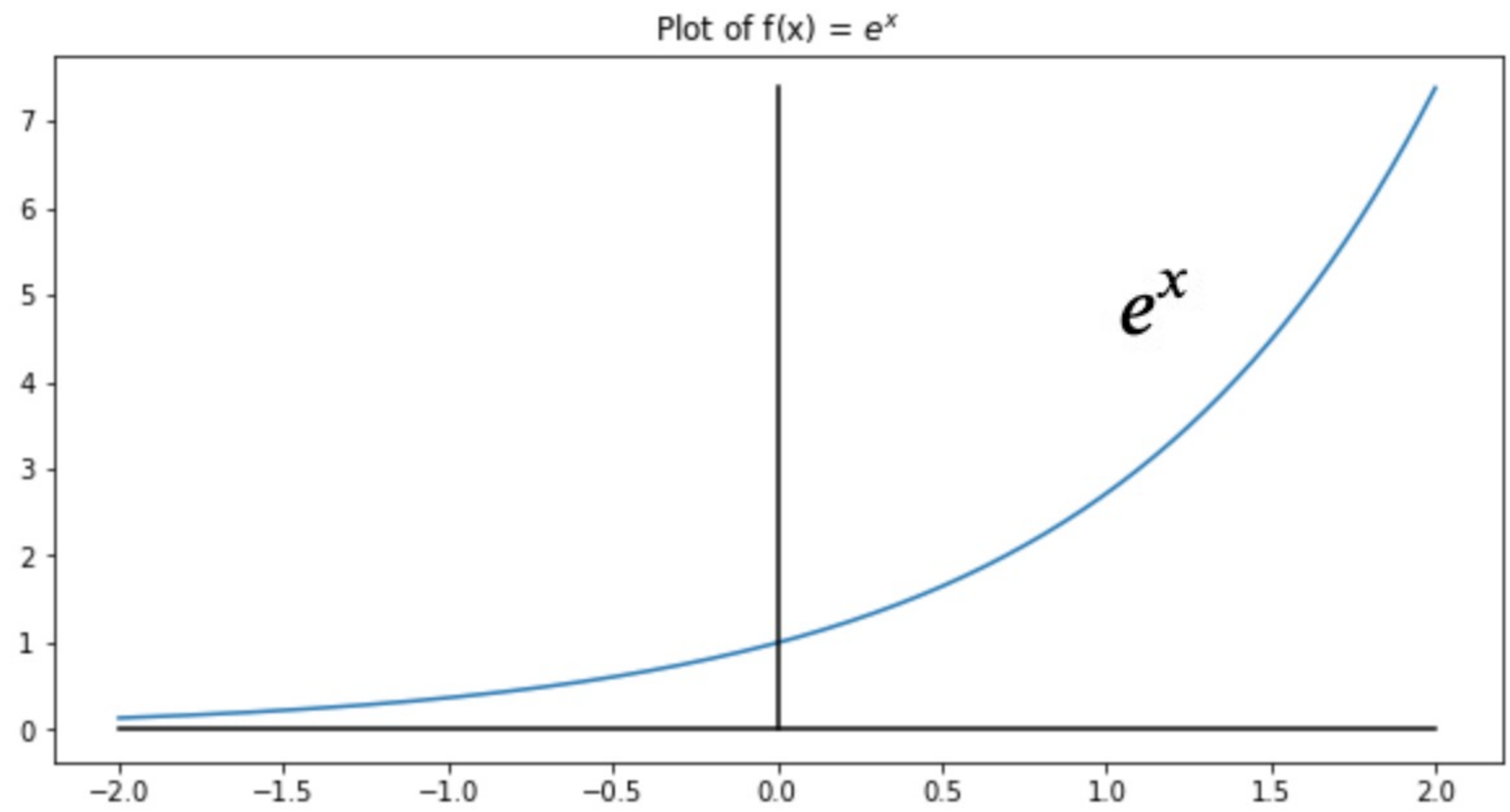


# Review: Exponential Function

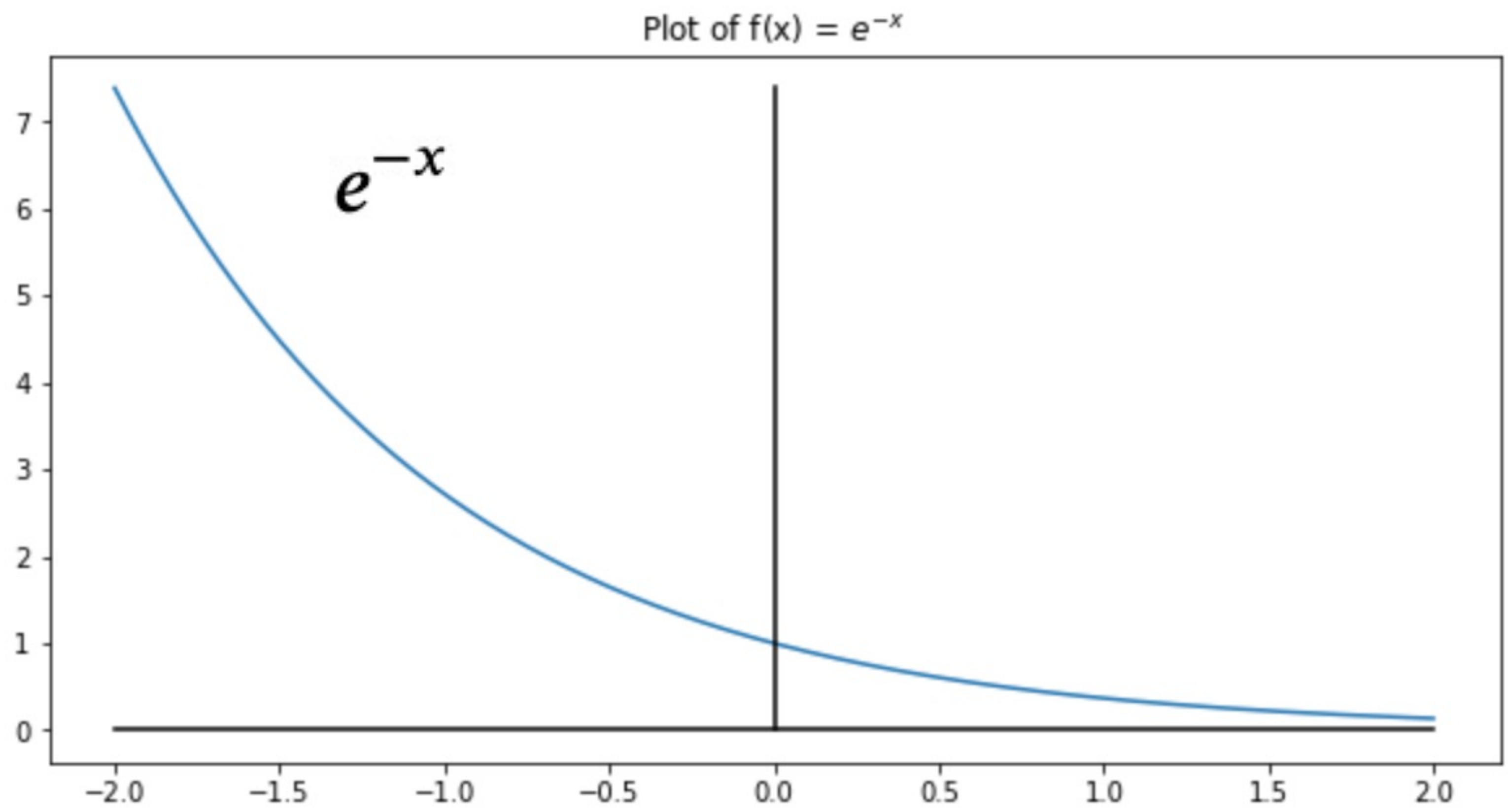
Since both of the continuous distributions we study (Normal and Exponential) use exponentials, let's think about this a bit....

Here is a graph of the exponential function $e^x$, where e = 2.71828... (Euler's Constant):



# Exponential Function

Here is a graph of $e^{-x}$, which flips the function around the Y axis:



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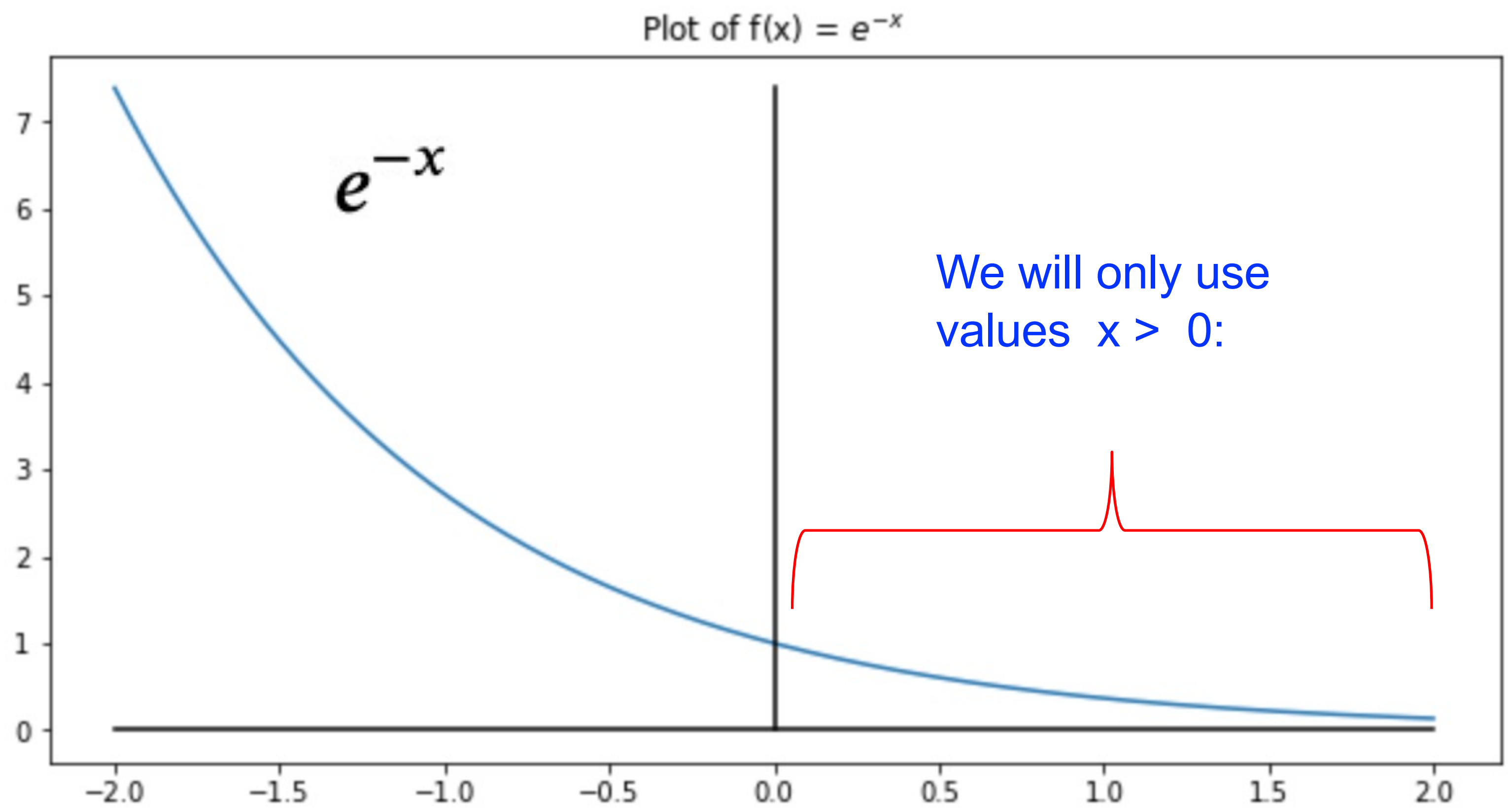


# Exponential Distribution

This is called the Exponential Distribution, and along with the Normal, is one of the most important continuous distributions in probability and statistics.

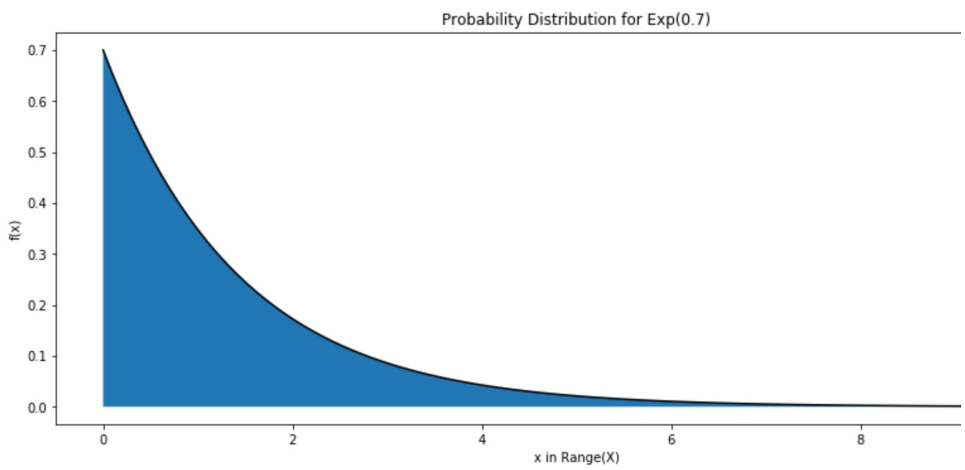
Formally, then, we say that Y is distributed according to the Exponential Distribution with rate parameter $\lambda$, denoted

$$Y \sim Exp(\lambda)$$

if

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$




and where $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$

# Exponential Distribution

## Exponential Distribution: Exp(λ)

**Motivation:** If we have a process in which events arrive (hence, c
the Exponential characterizes the inter-arrival time, e.g., "how lon

**Definition:** X ~ Exp(λ) if

$$Rng(X) = [0, \infty)$$
$$f(t) = \lambda e^{-\lambda t}$$
$$F(t) = 1.0 - e^{-\lambda t}$$

$$E(X) = \frac{1}{\lambda}$$
$$Var(X) = \frac{1}{\lambda^2}$$

$$P(X > t) = e^{-\lambda t}$$
$$P(X \leq t) = 1.0 - e^{-\lambda t}$$

where $e = 2.71828183 \ldots$ (Euler's constant).

## Geometrical Distribution: Geometric(p)

**Motivation:** This counts the number of Bernoulli trials until the first success occurs.

It can be viewed as a countable sequence of i.i.d. Bernoulli trials:

$$X_1, X_2, X_3, \ldots$$

where we return the smallest index $i$ for which $X_i = 1$.

**Definition:** X ~ Geometric(p) if

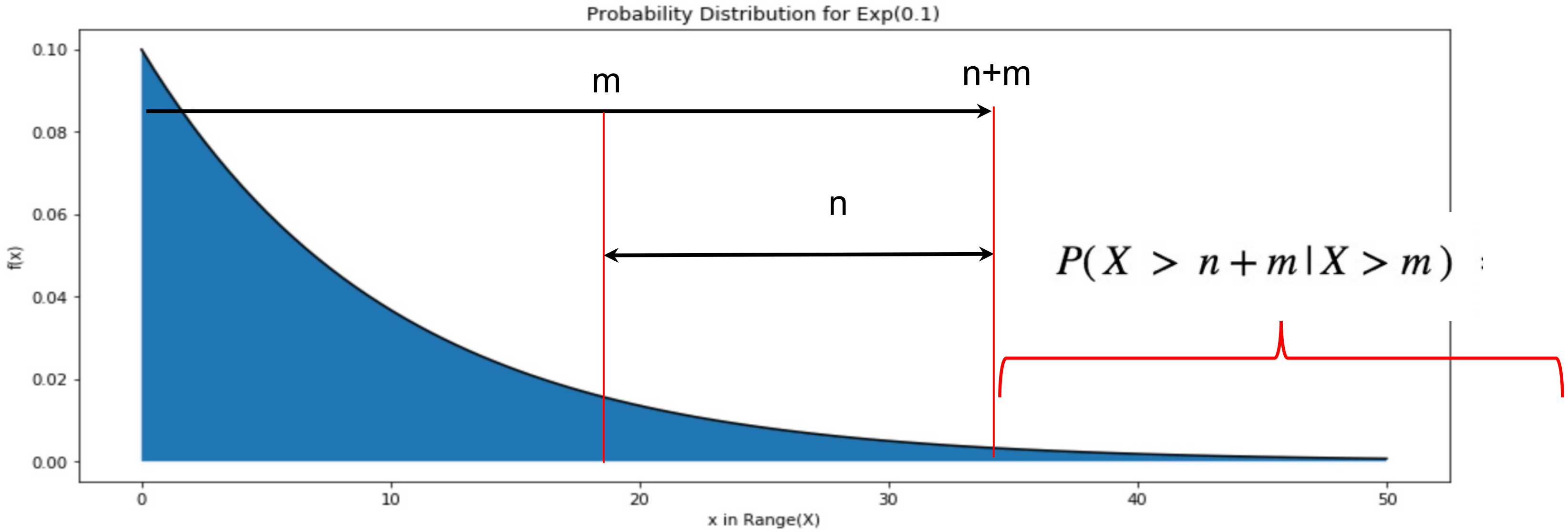
$$R_X = \{1, 2, \ldots\}$$
$$P_X(k) = (1-p)^{k-1} p$$

**Useful Formulae:**

$$E(X) = \frac{1}{p}$$
$$Var(X) = \frac{1-p}{p^2}$$

$$P(X > k) = (1-p)^k$$
$$P(X \leq k) = 1.0 - (1-p)^k$$

# Exponential Distribution: The Memoryless Property

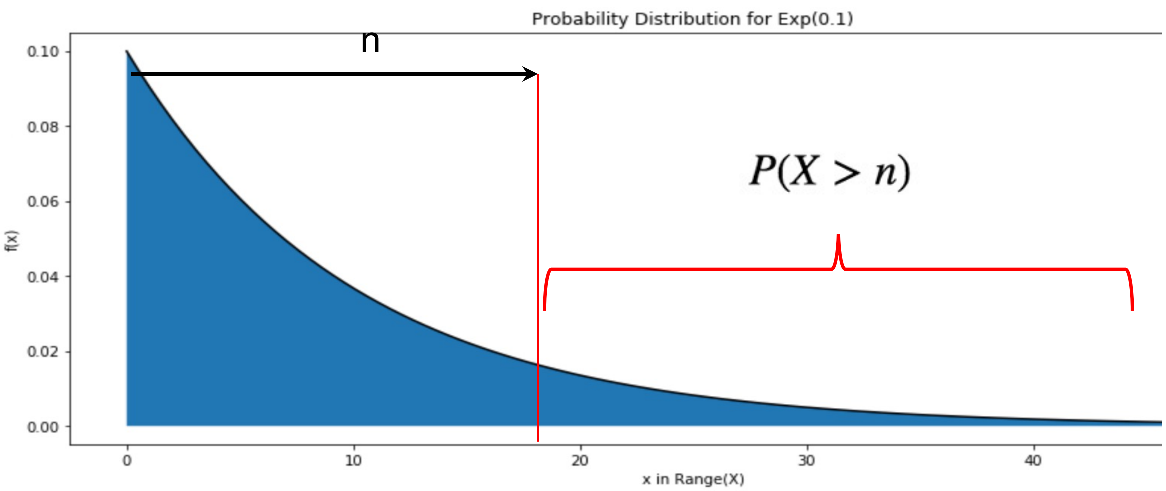



The exponential, like the geometric, has the memoryless property,

$$P(X > n+m \mid X > m) = P(X > n)$$

and the proof is the same!

$$\begin{aligned} P(X > n+m \mid X > m) &= \frac{P(X > n+m \text{ and } X > m)}{P(X > m)} \\ &= \frac{P(X > n+m)}{P(X > m)} \\ &= \frac{(1-p)^{(n+m)}}{(1-p)^m} \\ &= (1-p)^n \\ &= P(X > n) \end{aligned}$$



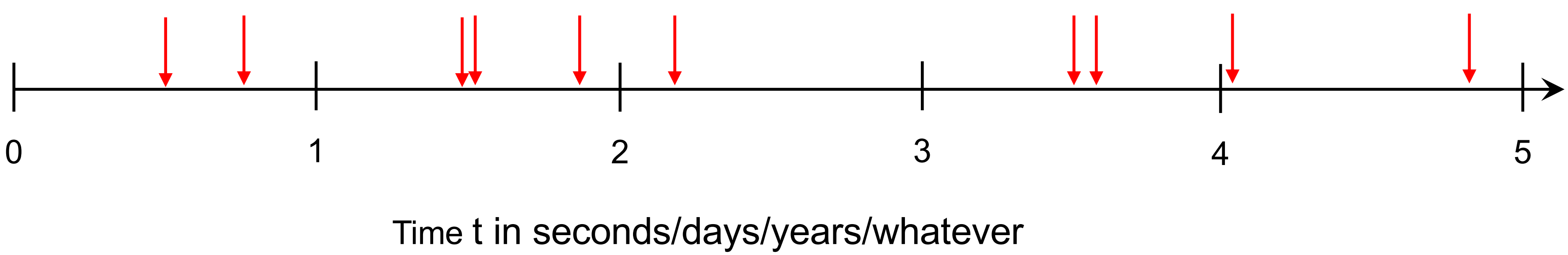
# Poisson Process

The Poisson Process concept captures an important way of thinking about events randomly occurring through time (or space)... Two things to remember are

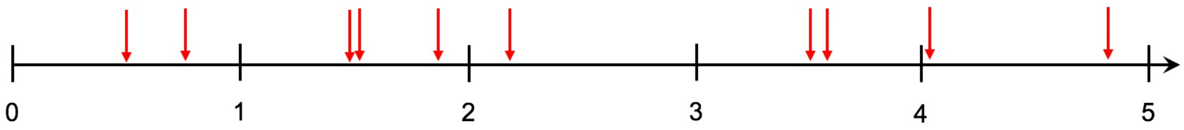
- **Events are discrete** (they happen or they don't – you can think of it as a Bernoulli trial with an outcome of success or failure), but
- **Time and space are continuous**..... the random behavior here is the time of an event.

When an event has happened we say it has arrived. You can think of this as a sequence of real numbers giving the **arrival time** of an event:

**Arrival Times** = { 0.4324... , 0.734 , 1.389... , 1.453... , 2.1546... , ... }

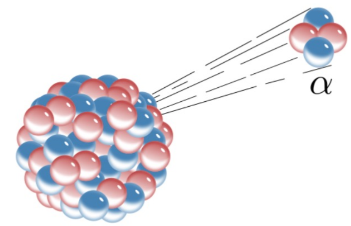


# Poisson Process




## Examples in the time domain:

- Sneezes in a classroom
- Alpha particles emitted from U 238
- Email arriving in my inbox
- Accidents at an intersection
- Earthquakes, volcanoes, asteroids, ...



### What if an asteroid hit the Earth?

BY MARSHALL BRAIN

UP NEXT »

**An illustration of an asteroid on its way to Earth. See more space dust images.**
PHOTOGRAPHER: ANDREUS AGENCY: DREAMSTIME.COM

An asteroid striking our planet -- it's the stuff of science fiction. Many movies and books have portrayed this possibility ("Deep Impact," "Armageddon," "Lucifer's Hammer," and so on).

An asteroid impact is also the stuff of science fact. There are obvious craters on Earth (and the moon) that show us a long history of large objects hitting the planet. The most famous asteroid ever is the one that hit Earth 65 million years ago. It's thought that this asteroid threw so much moisture and dust in to the atmosphere that it cut off sunlight, lowering temperatures worldwide and causing the extinction of the dinosaurs.

## Yellowstone volcano eruption: NASA to SAVE the world from supervolcano erupting

NASA scientists are creating an ambitious plan to prevent an explosion of a Yellowstone volcano that could even end human life by drilling a hole.

## THE REALLY BIG ONE

*An earthquake will destroy a sizable portion of the coastal Northwest. The question is when.*

By Kathryn Schulz

When the 2011 earthquake and tsunami struck Tohoku, Japan, Chris Goldfinger was two hundred miles away, in the city of Kashiwa, at an international meeting on seismology. As the shaking started, everyone in the room began to laugh. Earthquakes are common in Japan—that one was the third of the week—and the participants were, after all, at a seismology conference. Then everyone in the room checked the time.

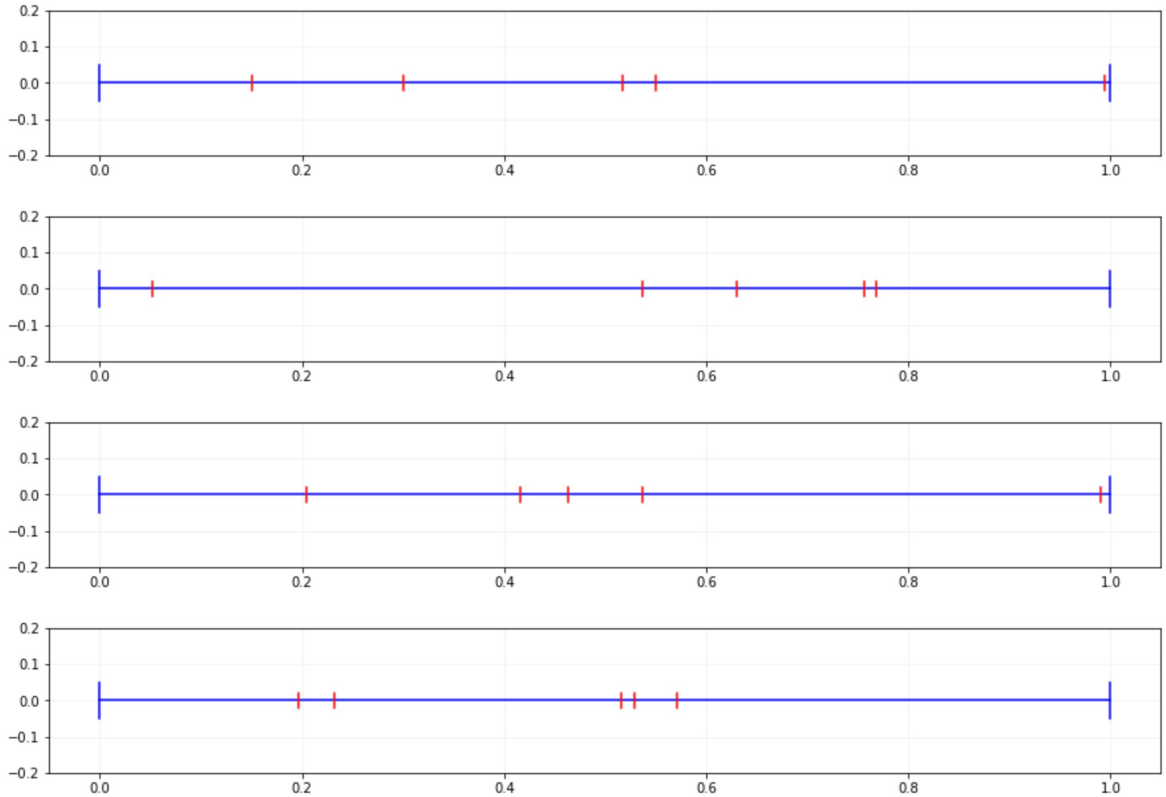
Seismologists know that how long an earthquake lasts is a decent proxy for its

*The next full-margin rupture of the Cascadia subduction zone will spell the worst natural disaster in the history of the continent.*

Every year The Federal Highway Administration reports approximately 2.5 Million intersection accidents. Most of these crashes involve left turns.

# Poisson Process

We can motivate the way a Poisson process is formally defined by considering what happens when we randomly generate arrivals in a unit interval. Suppose each trial of the experiment we generate 5 random numbers in the interval [0..1):



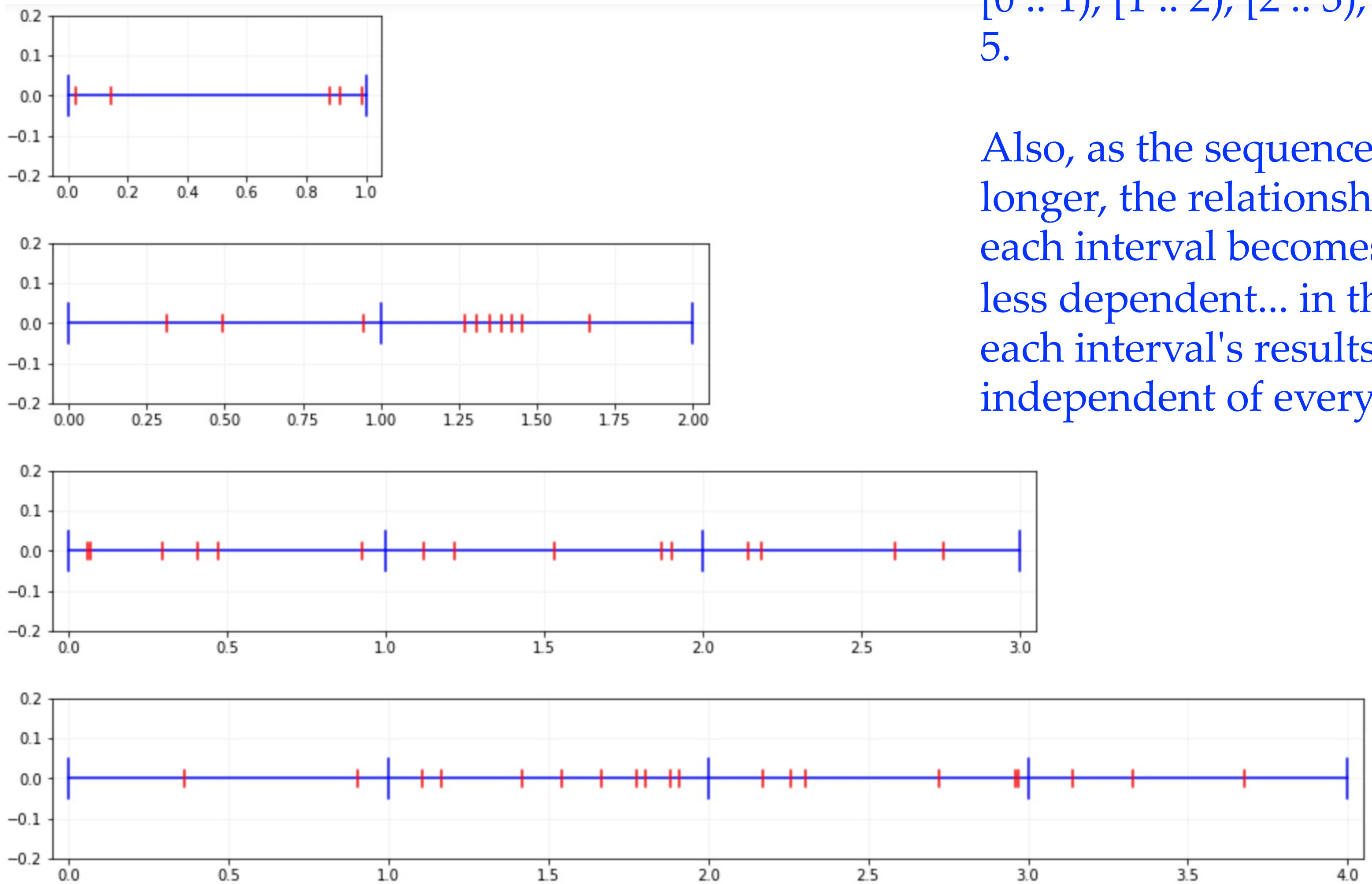

We know that the probability that a particular arrival occurs in the interval [0.0 .. 0.1) is 1/10; for [0.2 .. 0.5) is 0.3, and for any interval [a..b) it is (b-a).

The probability for any one arrival is equal to the length of the interval.

This is because the arrivals are randomly and uniformly distributed in the interval [0..1).

# Poisson Process

Now suppose we generate 5 random arrivals in [0 .. 1),  10 random arrivals in [0 .. 2), 15 arrivals in [0 .. 3),  20 in [0 .. 4) and so on, to infinity....




Since the arrivals are independent and distributed uniformly, the mean number of arrivals in each unit interval [0 .. 1), [1 .. 2), [2 .. 3), etc. is still 5.

Also, as the sequence gets longer, the relationship between each interval becomes less and less dependent... in the limit, each interval's results are independent of every other.

# Poisson Process

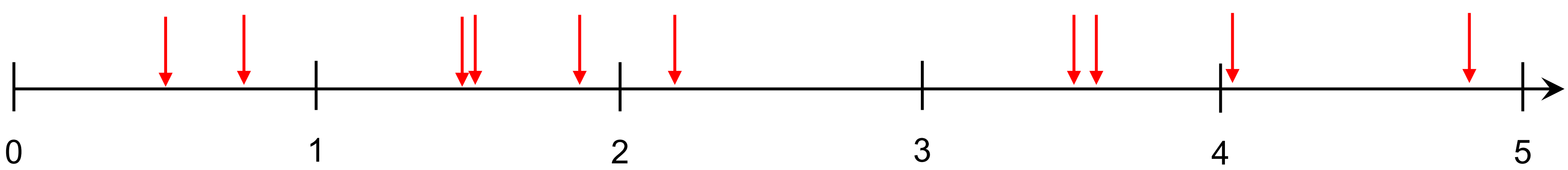
Formally, we have the following definition: suppose we have discrete events occurring through time as just described, and let us define a **Counting Random Variable**

$$N[s..t] = \text{the number of events arriving in the time interval } [s..t]$$

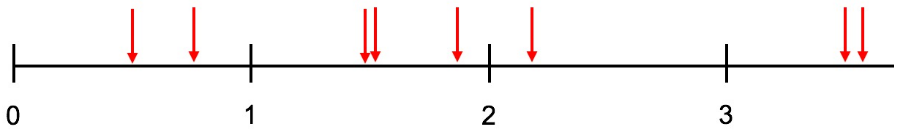
such that

1) The expected value of N[s..t]:
   a) is a fixed constant $\lambda$ over any unit interval anywhere in the sequence, and
   b) is proportional to the length (t – s) of the interval; in particular, for any two non-overlapping intervals of the same length, the expected number of occurrences in each is the same;
2) The number of arrivals in two non-overlapping intervals is independent; and
3) The probability of two events occurring at the same time is 0.

Then this random process is said to be a Poisson Process.

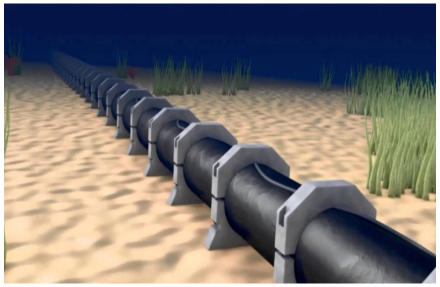


# Poisson Process

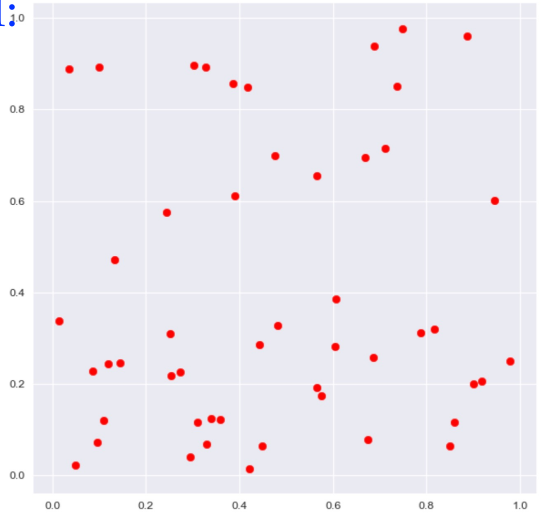



It is also possible that the continuous dimension is distance in space, in 1 dimension or more than 1. Examples include:

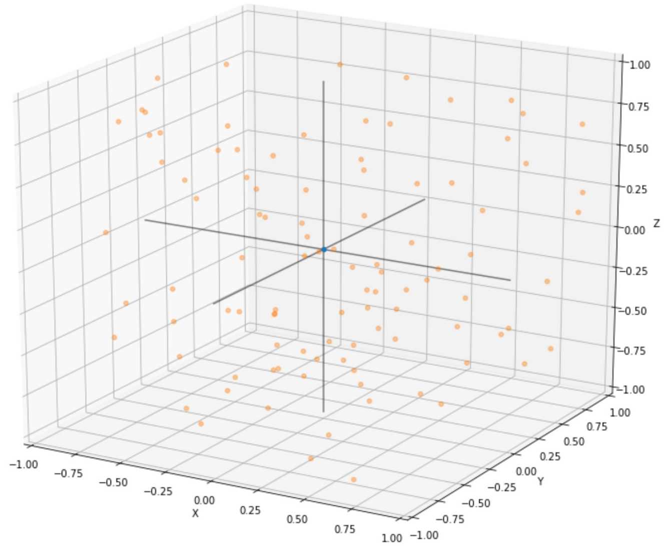
The occurrence of leaks in an undersea pipeline (1D):



Location of trees in a 1 square mile plot of land:



Location of supernovas in a given cubic gigaparsec volume of space in the last billion years:



**The important point is that events (discrete) occur along 1 or more (continuous) dimensions.**

# Poisson Random Variables (Discrete)

Suppose we have a Poisson Process and we fix the unit time interval we consider (say, 1 second or 1 year, etc.), where the mean number of arrivals in a unit interval is $\lambda$, and then each time we "poke" the random variable X we return N[0..1), N[1..2), N[2..3), etc.

Then we call X a Poisson Random Variable with rate parameter $\lambda$, denoted

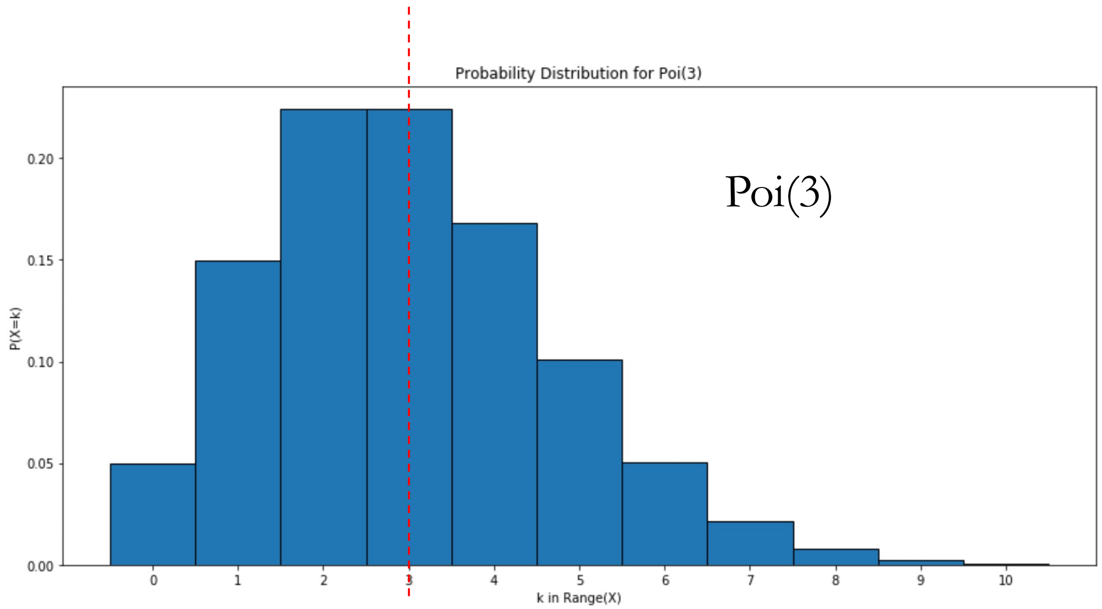
$$X \sim Poi(\lambda)$$

where

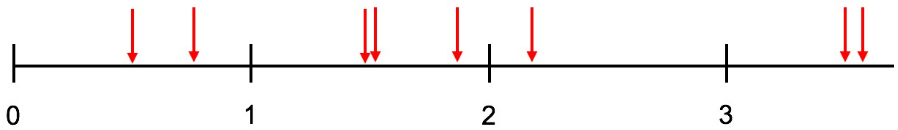
$$R_X = \{0, 1, 2, 3, \ldots\}$$

$$f_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

E(X) = Var(X) = $\lambda$



# Poisson Random Variables

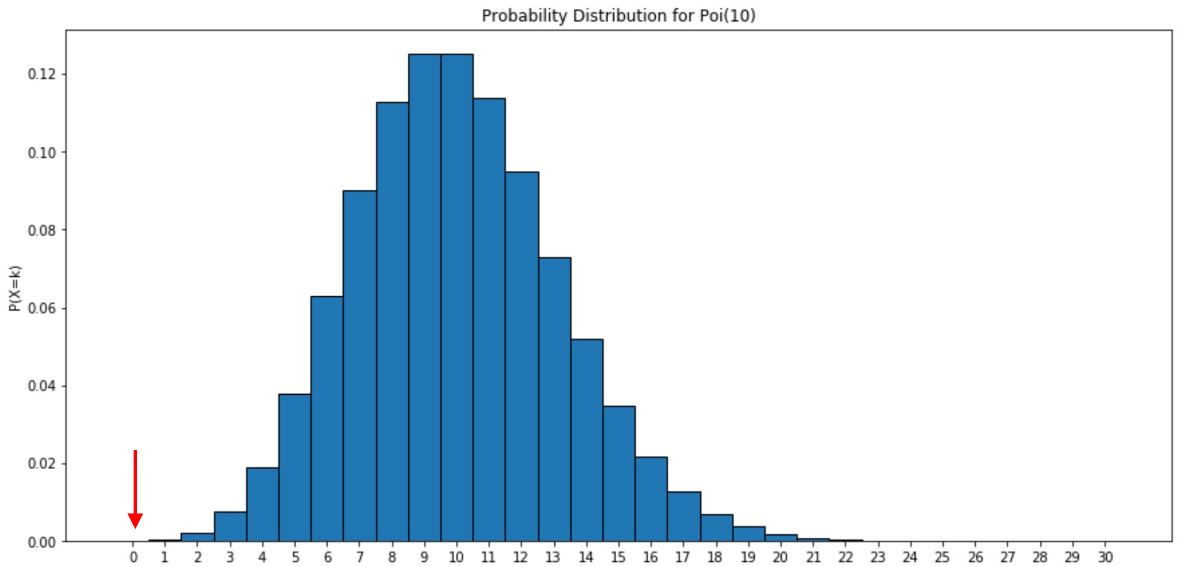



## Examples

Assume that arrivals of email in my Inbox are a Poisson Process with rate $\lambda = 10$ messages per hour. Then $X \sim Poi(10)$ returns the random number of emails which arrive within any particular hour.

What is the probability that I get no emails in the next hour?

$$P(X = 0) = f_X(0) = \frac{e^{-10}\lambda^0}{0!} = e^{-10} = 4.54 * 10^{-5}$$

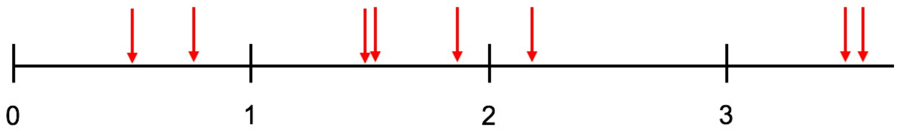



$$X \sim Poi(10)$$

$$R_X = \{0, 1, 2, 3, \dots\}$$

$$P(X = k) = f_X(k) = \frac{e^{-10}10^k}{k!}$$

# Poisson Random Variables

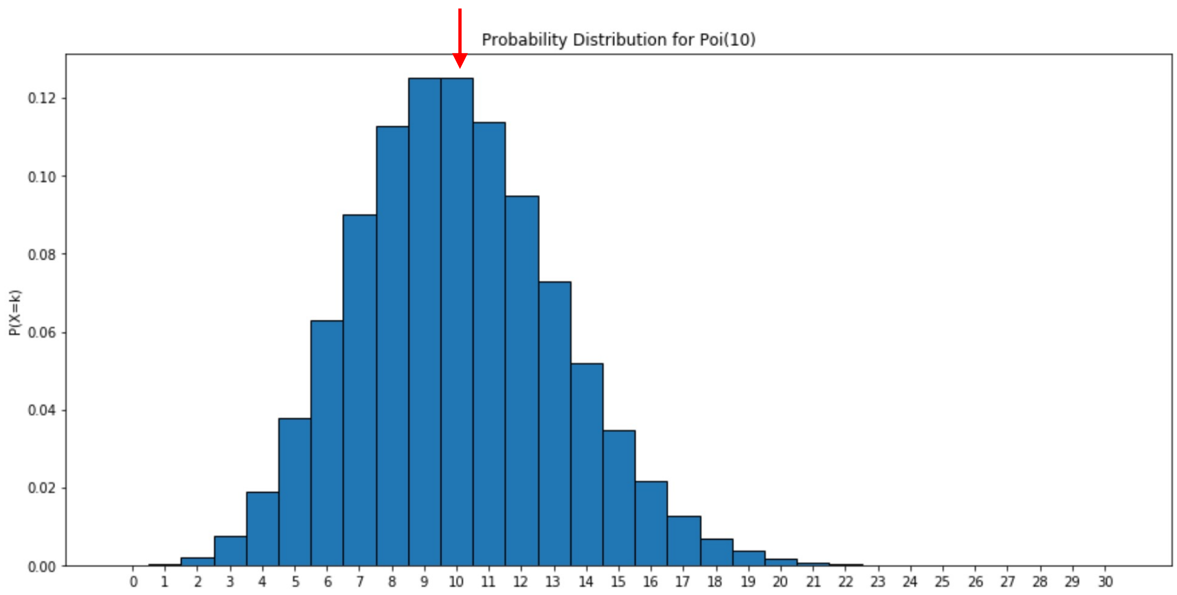



## Examples

Assume that arrivals of email in my Inbox are a Poisson Process with rate $\lambda = 10$ messages per hour. Then $X \sim Poi(10)$ returns the random number of emails which arrive within any particular hour.

What is the probability that I get exactly 10 emails in the next hour?

$$P(X = 10) = f_X(10) = \frac{e^{-10}\lambda^{10}}{10!} = 0.1251$$

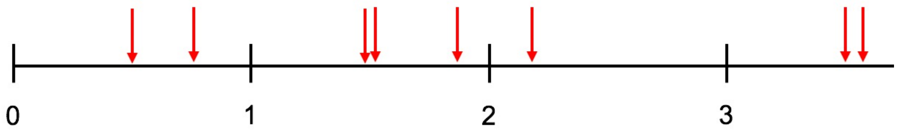



$$X \sim Poi(10)$$

$$R_X = \{0, 1, 2, 3, \dots\}$$

$$P(X = k) = f_X(k) = \frac{e^{-10}10^k}{k!}$$

# Poisson Random Variables

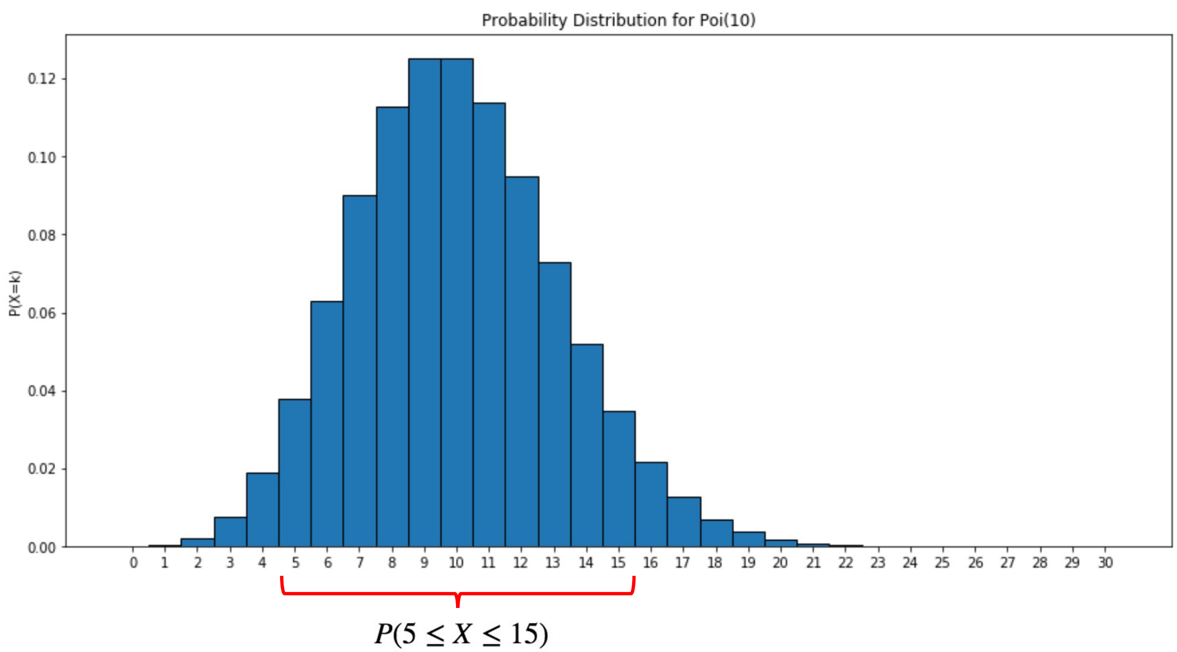



## Examples

Unfortunately there is no way to compute the CDF or ranges except by simply adding together all the individual values.

What is the probability that I get between 5 and 15 emails (inclusive) emails in the next hour?

$$P(5 \le X \le 15) = \sum_{k=5}^{15} \frac{e^{-10} 10^k}{k!} = 0.922$$




$$X \sim Poi(10)$$

$$R_X = \{0, 1, 2, 3, \dots\}$$

$$P(X = k) = f_X(k) = \frac{e^{-10} 10^k}{k!}$$

## Optional: The Waiting Time Paradox:

- **https://jakevdp.github.io/blog/2018/09/13/waiting-time-paradox/**

# Recall: Poisson Random Variables

Suppose we have a Poisson Process and we fix the unit time interval we consider (say, 1 second or 1 year, etc.), where the mean number of arrivals in a unit interval is $\lambda$, and then each time we "poke" the random variable X we return N[0..1], N[1..2], N[2..3], etc.

Then we call X a Poisson Random Variable with rate parameter $\lambda$, denoted

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where

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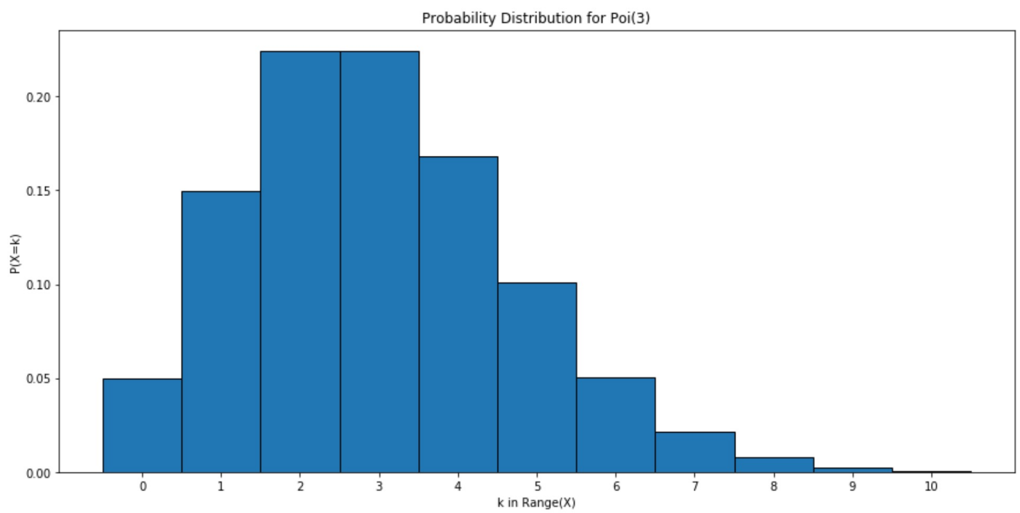
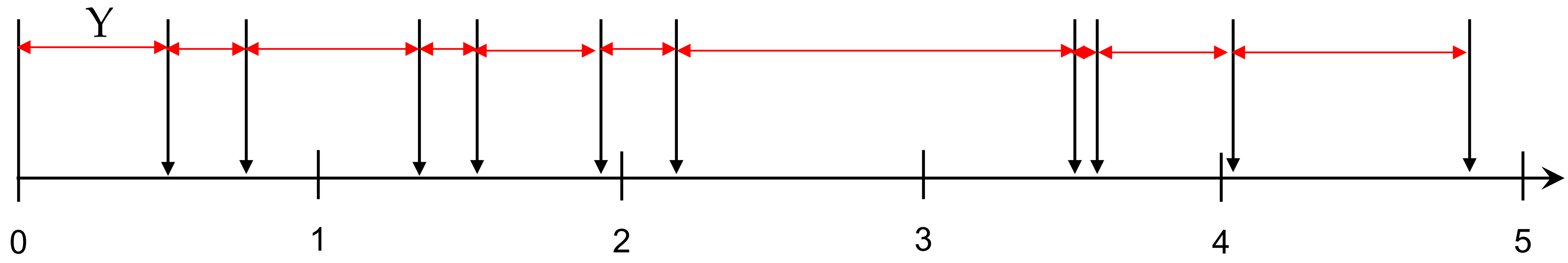


# Interarrival Times of a Poisson Process

Suppose we have a Poisson Process, and instead of counting the number of arrivals in each unit interval, we look at the interarrival times, i.e., the amount of time between each arrival.

Intuitively, this is a natural thing to think about: **How long before the next event?**




Let's define the random variable Y = "the arrival time of the first event."

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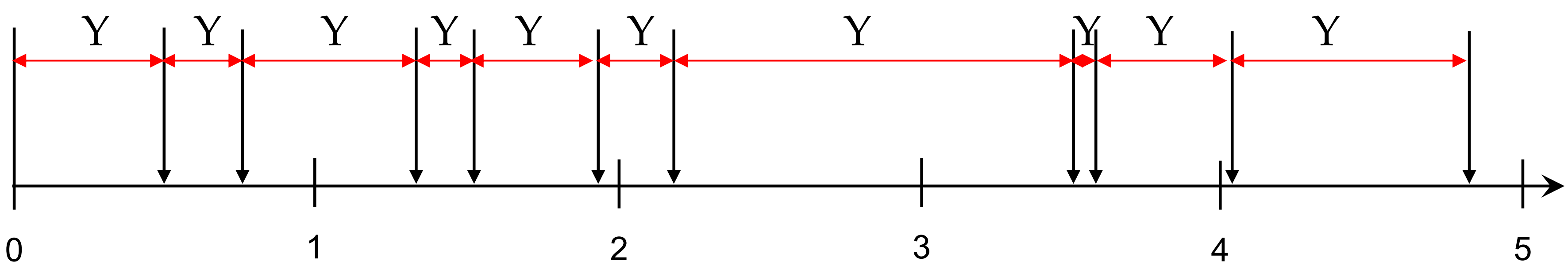


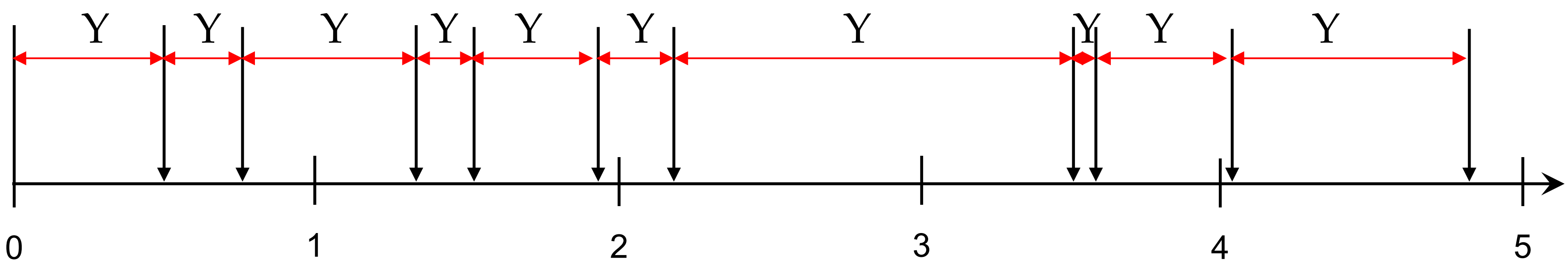

Let's define the random variable Y = "the arrival time of the first event."

In fact, because the arrivals are independent, at any time t, probabilistically the Poisson process starts all over again (the events don't remember the past!), so in fact:

Y = "the interarrival time between any two events"

Now the question is: **What is the distribution of Y?**

# Interarrival Times of a Poisson Process




What is the distribution of Y? Since

$$\lambda = E(N[0..1])$$

and the number of arrivals in an interval is proportional to its length, that is, E( N[0..2] ) = 2 * E( N[0..1] ), etc., then $\lambda \cdot t = E(N[0..t])$

and so the probability that there are exactly n arrivals by time t is

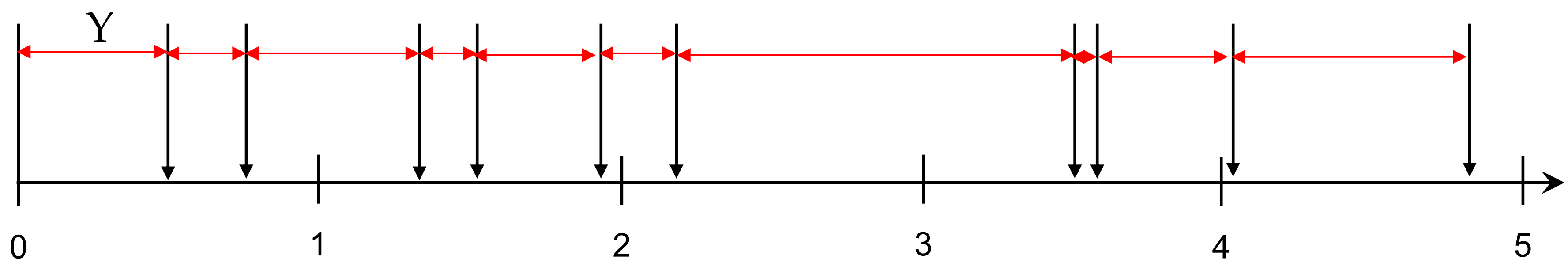
$$P(N[0..t] = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

and

$$P(Y > t) = P(N[0..t] = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$P(Y \le t) = 1 - e^{-\lambda t}$$

# Interarrival Times of a Poisson Process = Exponential Random Variable




What is the distribution of Y?

$$P(Y \leq t) = 1 - e^{-\lambda t}$$

Now, this is **the CDF of the Exponential**:

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Recall the derivative of exponential:

$$\frac{d\,e^{cx}}{dx} = ce^{cx}$$

and the chain rule:

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

and so if we take the derivative $0 - (-\lambda)e^{-\lambda t} = \lambda e^{-\lambda t}$ we get the **PDF of the Exponential:**

$$f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$